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THE SOLUTION OF INTEGRAL EQUATIONS WHICH ARISE IN PERIODIC PROBLEMS WITH MIXED BOUNDARY CONDITIONS†

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A two-parameter integral equation of the first kind with a difference periodic kernel, to which a wide range of periodic problems of the mechanics of continua with mixed boundary conditions can be reduced, is investigated. For the two main versions it is converted to a singular integral equation, which can be effectively solved using many well-known approximate methods. In a special case closed solutions of the initial equation are obtained. Antiplane contact problems for an elastic plane and a cylindrical layer are considered.

1. The following two-parameter integral equation of the first kind with a difference periodic kernel arises in periodic problems of the mechanics of continua and in other problems of mathematical physics [1–4]

$$\int_{-1}^{1} \varphi(\xi)K[\alpha(\xi-x)]d\xi = \pi f(x) \quad (|x|\leqslant 1) \tag{1.1}$$

$$K(y)=\frac{1}{2}\sum_{k=-\infty}^{\infty}\frac{L(\beta u_k)}{u_k}e^{iu_k y} \; (y=\alpha(\xi-x)) \tag{1.2}$$

Here α and β are dimensionless positive parameters, where $0<\alpha<\pi$ and $0<\beta<\infty$, the function $f(x)$ is specified and such that its first derivative for $|x|\leqslant 1$ satisfies the Hölder condition, and the function $L(\upsilon)$ is odd, continuous and does not vanish when $0<|\upsilon|<\infty$. Moreover, we have the following relations

$$L(|\upsilon|)=1+O(\upsilon^{-2}) \;\; (|\upsilon|\to\infty), \quad L(\upsilon)=A\upsilon+O(\upsilon^3)\; (\upsilon\to 0) \tag{1.3}$$

where A is a positive constant. As regards the quantities u_k, two fundamental cases are encountered (henceforth denoted by 1 and 2)

$$(1)\;\; u_k=k-\tfrac{1}{2}, \; (2)\;\; u_k=k \tag{1.4}$$

By virtue of (1.3) we can represent the function $L(\upsilon)$ in the form

$$\begin{aligned} &L(\upsilon)=\operatorname{th}A\upsilon+g(\upsilon) \\ &g(|\upsilon|)=O(\upsilon^{-2})(|\upsilon|\to\infty), \quad g(\upsilon)=O(\upsilon^3)\;\;(\upsilon\to 0) \\ &|g(\upsilon)|\leqslant\delta\;\;(0<|\upsilon|<\infty) \end{aligned} \tag{1.5}$$

where the quantity δ is usually small in practical problems.

Consider the following series (everywhere henceforth summation is carried out from $k=1$ to $k=\infty$)

$$M_i(y)=\sum \operatorname{th}\gamma u_k \sin u_k y \;\; (\gamma=\beta A) \tag{1.6}$$

In cases 1 ($i=1$) and 2 ($i=2$) this can, correspondingly, take the form ([5], formulae 1.441(2), 1.442(2) and 8.146(10, 11))

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$$M_1(y)=\frac{1}{2}\left[\operatorname{cosec}\frac{y}{2}-4\sum\frac{q^{2k-1}}{1+q^{2k-1}}\sin\left(k-\frac{1}{2}\right)y\right]=\mathrm{K}(e)F_1(u)$$

$$M_2(y)=\frac{1}{2}\left[\operatorname{ctg}\frac{y}{2}-4\sum\frac{q^{2k}}{1+q^{2k}}\sin ky\right]=\mathrm{K}(e)F_2(u)\quad (q=e^{-\gamma}) \tag{1.7}$$

$$F_1(u)=\frac{\operatorname{dn}u}{\operatorname{sn}u},\quad F_2(u)=\frac{\operatorname{cn}u}{\operatorname{sn}u}\quad\left(u=\frac{\mathrm{K}(e)y}{\pi}\right)$$

The quantity $e < 1$ is found from the transcendental equation

$$\pi\,\mathrm{K}(\sqrt{1-e^2})[\mathrm{K}(e)]^{-1}=\gamma \tag{1.8}$$

where $\mathrm{K}(e)$ is the complete elliptic integral of the first kind and $\operatorname{sn}u$, $\operatorname{cn}u$ and $\operatorname{dn}u$ are Jacobi elliptic functions.

We differentiate integral equation (1.1), (1.2) once with respect to x, and using relations (1.5)–(1.7) for cases 1 and 2 we write it in the form

$$\mu\int_{-1}^{1}\varphi(\xi)F_i[\mu(\xi-x)]d\xi=\pi f'(x)-\alpha\int_{-1}^{1}\varphi(\xi)G_i[\alpha(\xi-x)]d\xi \tag{1.9}$$

$$\mu=\pi^{-1}\mathrm{K}(e)\alpha,\quad G_i(y)=\sum g(\beta u_k)\sin u_k y$$

It can be shown, on the basis of the properties of $g(\upsilon)$, that the functions $G_i(y)$ satisfy the Hölder condition, when $|y| \leqslant 2\alpha$.

2. We will consider separately the cases of even versions of Eqs (1.9) ($\varphi(x)$ and $f(x)$ are even functions) and odd versions ($\varphi(x)$ and $f(x)$ are odd functions).

Note that

$$\begin{aligned}
&F_1[\mu(\xi-x)]-F_1[\mu(\xi+x)]=2\operatorname{sn}\mu x\operatorname{cn}\mu\xi\operatorname{dn}\mu x/\Delta\\
&F_1[\mu(\xi-x)]+F_1[\mu(\xi+x)]=2\operatorname{sn}\mu\xi\operatorname{cn}\mu x\operatorname{dn}\mu\xi/\Delta\\
&F_2[\mu(\xi-x)]-F_2[\mu(\xi+x)]=2\operatorname{sn}\mu x\operatorname{cn}\mu x\operatorname{dn}\mu\xi/\Delta\\
&F_2[\mu(\xi-x)]+F_2[\mu(\xi+x)]=2\operatorname{sn}\mu\xi\operatorname{cn}\mu\xi\operatorname{dn}\mu x/\Delta\\
&\Delta=\operatorname{sn}^2\mu\xi-\operatorname{sn}^2\mu x
\end{aligned} \tag{2.1}$$

Using (2.1) and taking into account the fact that $\mu < \mathrm{K}(e)$, while the functions $\operatorname{cn}\mathrm{K}x$ and $\operatorname{dn}\mathrm{K}x$ decrease monotonically from 1 to 0 as x increases from 0 to 1 [6], we reduce (1.9) to the form

$$\mu\int_{-1}^{1}\frac{\varphi(\xi)\operatorname{cn}\mu\xi}{\operatorname{sn}\mu\xi-\operatorname{sn}\mu x}d\xi=\frac{\pi f'(x)}{\operatorname{dn}\mu x}-\frac{\alpha}{\operatorname{dn}\mu x}\int_{-1}^{1}\varphi(\xi)G_i[\alpha(\xi-x)]d\xi \tag{2.2}$$

$$\mu\int_{-1}^{1}\frac{\varphi(\xi)\operatorname{dn}\mu\xi}{\operatorname{sn}\mu\xi-\operatorname{sn}\mu x}d\xi=\frac{\pi f'(x)}{\operatorname{cn}\mu x}-\frac{\alpha}{\operatorname{cn}\mu x}\int_{-1}^{1}\varphi(\xi)G_i[\alpha(\xi-x)]d\xi$$

The first equation of (2.2) holds for the even versions of Eq. (1.9) for $i = 1$ and the odd version of Eq. (1.9) for $i = 2$, while the second equation of (2.2) holds for the odd version of Eq. (1.9) for $i = 1$ and the even version of Eq. (1.9) for $i = 2$.

Taking into account once again the fact that $\mu < \mathrm{K}(e)$ while the function $\operatorname{sn}\mathrm{K}x$ increases monotonically from 0 to 1 as x increases from 0 to 1 [6], we introduce the new variables

$$\tau=\operatorname{sn}\mu\xi,\quad t=\operatorname{sn}\mu x,\quad c=\operatorname{sn}\mu \tag{2.3}$$

and we introduce the following function, inverse to $\operatorname{sn}u$

$$\xi=\frac{\operatorname{asn}\tau}{\mu},\quad x=\frac{\operatorname{asn}t}{\mu},\quad \operatorname{asn}t=\int_0^t\frac{dx}{\sqrt{(1-x^2)(1-e^2x^2)}} \tag{2.4}$$

(we have used the definition of the function sn u given in [15, formula 8.144(1)]).

Using (2.3) and (2.4) we can reduce Eq. (2.2) to the form

$$\int_{-c}^{c} \frac{\psi^{(j)}(\tau)}{\tau - t} d\tau = \pi h^{(j)}(t) - \int_{-c}^{c} \psi^{(j)}(\tau) H_i^{(j)}(\tau, t) d\tau \quad (|t| \leqslant c) \tag{2.5}$$

where $j = 1$ corresponds to the first of the equations in (2.2) while $j = 2$ corresponds to the second, and we have introduced the following notation

$$\frac{\varphi(\xi)}{\operatorname{dn}\mu\xi} = \psi^{(1)}(\tau), \quad \frac{f'(x)}{\operatorname{dn}\mu x} = h^{(1)}(t), \quad \frac{\varphi(\xi)}{\operatorname{cn}\mu\xi} = \psi^{(2)}(\tau), \quad \frac{f'(x)}{\operatorname{cn}\mu x} = h^{(2)}(t)$$

$$H_i^{(j)}(\tau, t) = \frac{\pi \tilde{H}_i^{(j)}(\tau, t)}{\mathrm{K}(e)} G_i\left[\frac{\pi}{\mathrm{K}(e)}(\operatorname{asn}\tau - \operatorname{asn} t)\right] \tag{2.6}$$

$$\tilde{H}_i^{(1)}(\tau, t) = \frac{1}{\sqrt{1 - e^2 t^2}\sqrt{1 - \tau^2}}, \quad \tilde{H}_i^{(2)}(\tau, t) = \tilde{H}_i^{(1)}(t, \tau)$$

It is important to note that $c < 1 < 1/e$ and the root singularities in the denominators of the expressions for $\tilde{H}_i^{(j)}(\tau, t)$ lie outside the ranges of definition and integration in (2.5).

3. Any of the well-known approximate methods [4, 7–11] can be used to solve the singular integral equation of the first kind (2.5). Since they are all in some way based on exact inversion of the principal singular operator on the left-hand side of (2.5), for small δ in (1.5) their efficiency will be extremely high for any values of the parameters α and β.

Taking the above properties of the functions $f(x)$ and $G_i(y)$ into account, it can be proved [4] that if a solution of Eq. (2.5) exists for specified values of α and β in the class of functions for which the integral

$$\int_{-c}^{c} |\psi^{(j)}(\tau)|^p d\tau \quad (0 < p < 2) \tag{3.1}$$

converges, then this solution can be represented in general in the form

$$\psi^{(j)}(t) = \Psi^{(j)}(t)(c^2 - t^2)^{-1/2} \tag{3.2}$$

where the function $\Psi^{(j)}(t)$ satisfies the Hölder condition when $|t| \leqslant c$.

If the function $f(x)$ and, consequently, the function $\varphi(x)$ in Eq. (1.1) have both even and odd parts, i.e.

$$f(x) = f_+(x) + f_-(x), \quad \varphi(x) = \varphi_+(x) + \varphi_-(x) \tag{3.3}$$

where the plus subscript denotes even parts while the minus subscript denotes odd parts, solving Eqs (2.5) we obtain for case 1

$$\varphi(x) = \operatorname{dn}\mu x \psi_+^{(1)}(\operatorname{sn}\mu x) + \operatorname{cn}\mu x \psi_-^{(2)}(\operatorname{sn}\mu x) \tag{3.4}$$

and for case 2

$$\varphi(x) = \operatorname{cn}\mu x \psi_+^{(2)}(\operatorname{sn}\mu x) + \operatorname{dn}\mu x \psi_-^{(1)}(\operatorname{sn}\mu x) \tag{3.5}$$

where $\Psi_+^{(j)}(t)$ and $\Psi_-^{(j)}(t)$ are defined as the solutions corresponding to $f'_+(x)$ and $f'_-(x)$.

Note that the functions $\Psi_+^{(j)}(t)$ are found from the singular integral equations (2.5) up to terms

$$C^{(j)}(c^2 - t^2)^{-1/2} \tag{3.6}$$

where $C^{(j)}$ are arbitrary constants which must be defined from the requirement that the solutions (3.4) and (3.5) of Eq. (2.5) also satisfy the initial equation (1.1) (undifferentiated with respect to x), for example, at the point $x = 0$. Note that the point $x = 0$ was only chosen for simplicity, and one can take any other point in the section $|x| \leqslant 1$, because Eqs (1.9) $\to$ (2.2) $\to$ (2.5) differ from (1.1) solely in the single operation of differentiation.

Putting $x = 0$ in (1.1) and using formulae (2.4) and (2.5) from [9], which in the notation employed here have the form

$$\sum \frac{\operatorname{th}\gamma(k-\frac{1}{2})}{k-\frac{1}{2}}\cos\left(k-\frac{1}{2}\right)y = \frac{1}{2}\ln\frac{1+\operatorname{cn}u}{1-\operatorname{cn}u} \tag{3.7}$$

$$\frac{1}{2}\gamma + \sum\frac{\operatorname{th}\gamma k}{k}\cos ky = \frac{1}{2}\ln\frac{1+\operatorname{dn}u}{1-\operatorname{dn}u}$$

(note that (3.7) can be obtained by integrating (1.6) and (1.7) using formula 5.135(3, 6) from [5]; in [9] they are given incorrectly—instead of th there should be tg), after some algebra, taking (2.6) and (3.7) into account, we obtain the following relation for determining $C^{(j)}$

$$\int_{-c}^{c} \psi_{+}^{(i)}(\tau)[P_i(\tau)+R_i(\tau)]d\tau = \pi\mu f(0) \tag{3.8}$$

where

$$P_1(\tau) = \frac{1}{2\sqrt{1-\tau^2}}\ln\frac{1+\sqrt{1-\tau^2}}{1-\sqrt{1-\tau^2}}, \quad P_2(\tau) = P_1(e\tau)$$

$$R_i(\tau) = \tilde{R}_i(\tau)Q_i\left[\frac{\pi}{\mathrm{K}(e)}\operatorname{asn}\tau\right]$$

$$\tilde{R}_1(\tau) = \frac{1}{\sqrt{1-\tau^2}}, \quad \tilde{R}_2(\tau) = \tilde{R}_1(e\tau) \tag{3.9}$$

$$Q_i(y) = \sum\frac{g(\beta u_k)}{u_k}\cos u_k y \quad (Q_i'(y) = -G_i(y))$$

In a number of problems the function $f(x)$ in (1.1), for specified α and β, is only defined up to the linear part $c_0 + c_1x$. To obtain c_0 and c_1 we need additional conditions, which are usually the following

$$\varphi(\pm 1) = 0 \tag{3.10}$$

4. If $g(\upsilon) \equiv 0$ in (1.5), Eq. (2.5) degenerates into a classical singular integral equation of the first kind with a Cauchy kernel, which can be solved in closed form (see, for example, [4]). Then, we have for the even version of case 1

$$\varphi_{+}(x) = \frac{\operatorname{dn}\mu x}{\pi X(x)}\left[P - \mu\int_{-1}^{1}\frac{X(\xi)f_{+}'(\xi)\operatorname{cn}\mu\xi}{W(\xi,\ x)}d\xi\right] \tag{4.1}$$

$$X(x) = \sqrt{c^2 - \operatorname{sn}^2\mu x}, \quad W(\xi,\ x) = \operatorname{sn}\mu\xi - \operatorname{sn}\mu x$$

The additional relation (3.8), which now serves to determine the constant P in (4.1), takes the form

$$\int_{-1}^{1}\frac{\varphi_{+}(\xi)\operatorname{cn}\mu\xi}{Y(\xi)}\ln\frac{1+Y(\xi)}{1-Y(\xi)}d\xi = 2\pi f_{+}(0), \quad Y(x) = \sqrt{1-\operatorname{sn}^2\mu x} \tag{4.2}$$

For condition (3.10) we have the following equation for the even version of case 1

$$\varphi_{+}(x) = -\frac{\mu\operatorname{dn}\mu x}{\pi}X(x)\int_{-1}^{1}\frac{f_{+}'(\xi)\operatorname{cn}\mu\xi}{X(\xi)W(\xi,\ x)}d\xi \tag{4.3}$$

Relation (4.2) remains true here, while the additional conditions (3.10) for determining c_0 take the form

$$P+\mu\int_{-1}^{1}\frac{f_+'(\xi)\operatorname{sn}\mu\xi\operatorname{cn}\mu\xi}{X(\xi)}d\xi=0 \tag{4.4}$$

For the odd version of case 2 we must put $P = 0$ and replace $f_+'(\xi)$ by $f_-'(\xi)$ in (4.1). Under conditions (3.10), for the odd version of case 2, we must replace $f_+'(\xi)$ by $f_-'(\xi)$ in (4.3), while the additional conditions (3.10) for determining c_1 take the form

$$\int_{-1}^{1}\frac{f_-'(\xi)\operatorname{cn}\mu\xi}{X(\xi)}d\xi=0 \tag{4.5}$$

For the even version of case 2 we have

$$\varphi_+(x)=\frac{\operatorname{cn}\mu x}{\pi X(x)}\left[P-\mu\int_{-1}^{1}\frac{X(\xi)f_+'(\xi)\operatorname{dn}\mu\xi}{W(\xi,\ x)}d\xi\right] \tag{4.6}$$

and the additional relation (3.8) takes the form

$$\int_{-1}^{1}\frac{\varphi_+(\xi)\operatorname{dn}\mu\xi}{Z(\xi)}\ln\frac{1+Z(\xi)}{1-Z(\xi)}d\xi=\pi\mu f_+(0),\quad Z(x)=\sqrt{1-e^2\operatorname{sn}^2\mu x} \tag{4.7}$$

For conditions (3.10) we have the following equation for the even version of case 2

$$\varphi_+(x)=-\frac{\mu\operatorname{cn}\mu x}{\pi}X(x)\int_{-1}^{1}\frac{f_+'(\xi)\operatorname{dn}\mu\xi}{X(\xi)W(\xi,\ x)}d\xi \tag{4.8}$$

Relation (4.7) remains valid here, while the additional conditions (3.10) take the form

$$P+\mu\int_{-1}^{1}\frac{f_+'(\xi)\operatorname{sn}\mu\xi\operatorname{dn}\mu\xi}{X(\xi)}d\xi=0 \tag{4.9}$$

For the odd version of case 1 we must put $P = 0$ and replace $f_+'(\xi)$ by $f_-'(\xi)$ in (4.6). Under conditions (3.10) for the odd versions of case 1 we must replace $f_+'(\xi)$ by $f_-'(\xi)$ in (4.8), and the additional conditions (3.10) take the form

$$\int_{-1}^{1}\frac{f_-'(\xi)\operatorname{dn}\mu\xi}{X(\xi)}d\xi=0 \tag{4.10}$$

Note that when $f_+(\xi)\equiv f_+=\text{const}$, taking into account relations (3.152(7)) and (4.317(10)) from [5], for the even versions of cases 1 and 2 we have the following relations from formulae (4.1), (4.2), (4.6) and (4.7) [12]

$$\begin{gathered}\varphi_+(x)=\frac{\mu f_+\operatorname{dn}\mu x}{\mathrm{K}(\sqrt{1-c^2})X(x)},\quad N_0=\frac{2f_+\mathrm{K}(c)}{\mathrm{K}(\sqrt{1-c^2})}\\ \varphi_+(x)=\frac{\mu f_+\operatorname{cn}\mu x}{\mathrm{K}(\sqrt{1-e^2c^2})X(x)},\quad N_0=\frac{2f_+\mathrm{K}(ec)}{\mathrm{K}(\sqrt{1-e^2c^2})}\end{gathered} \tag{4.11}$$

where N_0 is an integral characteristic, defined by the formula

$$N_0=\int_{-1}^{1}\varphi_+(\xi)d\xi \tag{4.12}$$

5. We will consider the antiplane problem of the deformation of an elastic layer of thickness h, clamped along the base, by a periodic system of similar strip punches. Suppose the period is equal to $2b$, the width of a single punch is $2a(a < b)$ and there is complete adhesion between the punches and the upper surface of the layer.

If the punches are shifted along the generatrices by tangential forces T, directed alternately on different

Table 1

α	$\beta=2$	4	8
$\pi/9$	0.369	0.267	0.174
$2\pi/9$	0.490	0.326	0.197
$\pi/3$	0.599	0.372	0.213
$4\pi/9$	0.701	0.410	0.225

sides, the problem can be reduced to case 1 of integral equation (1.1), (1.2), and if the punches are shifted on one side, the problem can be reduced to case 2 of integral equation (1.1), (1.2) [1, 4]. Then, in the equation we have

$$\varphi_+(x)=\frac{\tau(ax)}{G},\quad L(\upsilon)=\operatorname{th}\upsilon,\quad \alpha=\frac{\pi a}{b},\quad \beta=\frac{\pi h}{b},\quad f(x)\equiv f_+=\frac{\varepsilon}{a} \tag{5.1}$$

where $\tau(\eta)$ is the contact shear stress, G is the shear modulus, and ε is the value of the displacement of each punch along the generatrix due to the force T applied to it.

The solution of this problem for cases 1 and 2 will obviously be given by formula (4.11), in which e is defined from (1.8) with $\gamma=\beta$, and $N_0=T/(Ga)$.

Consider the antiplane problem of the deformation of an elastic tube, clamped along the external boundary, by a cylindrical strip punch. Suppose the external and internal radii of the tube are a and b, respectively, there is complete adhesion between the punch and the inner surface of the tube, and the angle of contact of the punch with the surface of the tube is $2\alpha_0$. The punch is moved along its generatrix by a tangential force T.

This problem reduces to cases 2 of integral equation (1.1), (1.2) [13]. Here in the equation we have

$$\varphi_+(x)=\frac{\tau(\alpha x)}{G},\quad L(\upsilon)=\operatorname{th}\upsilon,\quad \alpha=\alpha_0,\quad \beta=\ln\frac{b}{a},\quad f(x)\equiv f_+=\frac{\varepsilon}{\alpha_0 a} \tag{5.2}$$

The solution of the problem is given by the two successive formulae (4.11) in which e is found from (1.8) with $\gamma=\beta$, while $N_0=T/(G\alpha_0 a)$. Table 1 gives values of the resistance coefficient $V=T/(G\varepsilon)$ for a number of values of α and β.

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